Including explicit isometries in lattices in $\mathrm{SO}(n,1)$

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Motivation and background

Motivating Question

Question: Can one construct lattices in $\mathrm{Isom}(\mathbb{H}^d)$ that contain "interesting" isometries?
(Here, a discrete subgroup Γ is a lattice if $\mathrm{vol}(\Gamma\backslash\mathbb{H}^d) < \infty$.)

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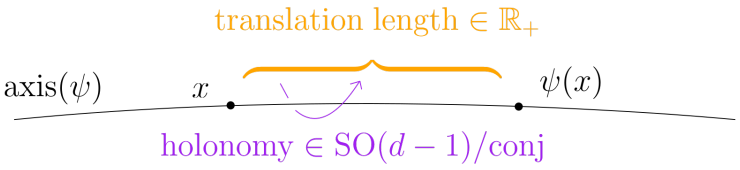
Goal: Construct an explicit family of loxodromic isometries that are contained in some lattice.

Loxodromic isometries of $\mathbb{H}^d$

Let $\psi \in \mathrm{Isom}(\mathbb{H}^d)$ be a loxodromic isometry.



complex translation length $\in \mathbb{R}_+ \times SO(d-1)/\text{conj}$.

Hyperboloid model for $\mathbb{H}^d$

Set $q(x) = x_0^2 + ... + x_{d-1}^2 - x_d^2$. Then, we may identify

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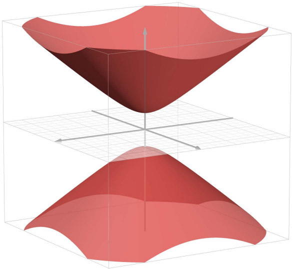

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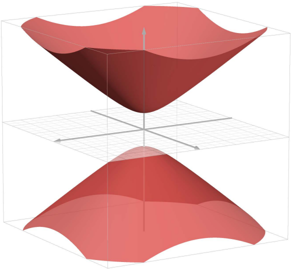

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In this model, we identify $\mathrm{Isom}^+(\mathbb{H}^d) := \mathrm{SO}^+(d,1)$ where

$$\mathrm{SO}^+(d,1) = \{A \in \mathrm{SL}_{d+1}(\mathbb{R}) \mid q(Ax) = q(x)\ \forall x \ \text{ and } \ A(\mathbb{H}^d) = \mathbb{H}^d\}.$$

Existing constructions of lattices

Theorem (Borel–Harish-Chandra 1962)

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$\mathrm{SO}^+(d,1;\mathbb{Z})$ *is a lattice in* $\mathrm{Isom}^+(\mathbb{H}^d)$.

In particular, every $\psi \in \mathrm{SO}^+(d,1;\mathbb{Z})$ is contained in a lattice.

Existing constructions of lattices

Theorem

For all $d \geq 2$ and $\varepsilon > 0$, there exists a torsion free lattice in $\mathrm{Isom}^+(\mathbb{H}^d)$ containing a loxodromic with translation length at most ε.

Equivalently, for all $d \geq 2$, there are finite volume hyperbolic d-manifolds containing arbitrarily short closed geodesics.

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- $d = 2$: Teichmuller theory
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Remark: For $d \geq 4$, the loxodromics constructed Agol and Belolipetsky-Thomson all have *trivial holonomy*.

Main results

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Theorem (Huang-Z. '26)

For every loxodromic $\psi \in \mathrm{SO}^+(d,1;\mathbb{Q})$*, there is a torsion free lattice* $\Gamma \leq \mathrm{SO}^+(d,1) = \mathrm{Isom}^+(\mathbb{H}^d)$ *containing* ψ*.*

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Remarks:

- Γ can be constructed so that ψ has minimal translation length.
- The pair $(q, \mathbb{Q})$ can be replaced with any "admissible" pair (q', k). In particular, a similar statement produces **cocompact** lattices.

Density corollary

Corollary

For every $d \geq 3$*, a dense subset of* $\mathbb{R}_+ \times \mathrm{SO}(d-1)/\mathrm{conj}$ *is realized by the complex lengths of closed geodesics appearing in finite volume hyperbolic d-manifolds.*

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Proof: $\mathrm{SO}^+(n, 1; \mathbb{Q})$ is dense in $\mathrm{Isom}^+(\mathbb{H}^d)$. $\square$

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Proof: $\mathrm{SO}^+(n, 1; \mathbb{Q})$ is dense in $\mathrm{Isom}^+(\mathbb{H}^d)$. □

Remark: For $d = 3$, this follows from work of Brooks (1986).

An explicit example

Infinite order holonomy (Explicit example)

Corollary

For every $d \geq 3$ *and* $\varepsilon > 0$, *there exists a lattice in* $\mathrm{Isom}^+(\mathbb{H}^d)$ *that contains a loxodromic isometry that has translation length less than* ε *and infinite order holonomy.*

Infinite order holonomy (Explicit example)

Corollary

For every $d \geq 3$ and $\varepsilon > 0$, there exists a lattice in $\mathrm{Isom}^+(\mathbb{H}^d)$ that contains a loxodromic isometry that has translation length less than ε and infinite order holonomy.

Proof: For each $k \geq 2$, set

$$\psi_k := \begin{pmatrix} 3/5 & -4/5 & & & \\ 4/5 & 3/5 & & & \\ & & I_{d-3} & & \\ & & & \frac{k^2+1}{k^2-1} & \frac{2k}{k^2-1} \\ & & & \frac{2k}{k^2-1} & \frac{k^2+1}{k^2-1} \end{pmatrix}$$

where I_{d-3} is the $(d-3) \times (d-3)$ identity matrix.

Infinite order holonomy (Explicit example)

A direct computation shows that each $\psi_k \in \mathrm{SO}^+(d,1;\mathbb{Q})$. Geometrically, ψ_k is a loxodromic satisfying

- $\mathrm{axis}(\psi_k) = \mathbb{H}^d \cap x_{d-1}x_d$ plane,
- $\mathrm{translation\ length}(\psi_k) = \log(\lambda_k)$, where

$$\lambda_k = \frac{k+1}{k-1},$$

- $\mathrm{holonomy}(\psi_k)$ is a rotation by $\arcsin(4/5) \notin 2\pi \cdot \mathbb{Q}$.

Choose k sufficiently large such that $\log(\lambda_k) < \varepsilon$, and apply the main theorem to produce a lattice containing ψ_k. □

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Proof sketch:

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Proof sketch: Fix a loxodromic $\psi \in \mathrm{SO}^+(d,1;\mathbb{Q})$. Set

$$\Gamma_1 \leq \mathrm{SO}^+(d,1;\mathbb{Z}), \qquad \text{finite index, torsion free.}$$

Recall that Γ_1 is a lattice, by Borel–Harish-Chandra.

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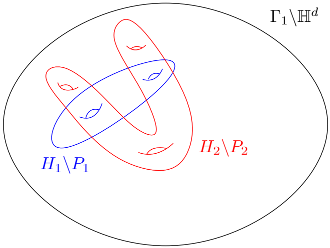
Choose disjoint rational hyperplanes $P_1, P_2 \subset \mathbb{H}^d$ such that

$$P_2 = \psi(P_1).$$

By rationality, P_1 and P_2 project to finite area totally geodesic hypersurfaces in $\Gamma_1 \backslash \mathbb{H}^d$.

Proof of main theorem

Inside $\Gamma_1 \backslash \mathbb{H}^d$ ...



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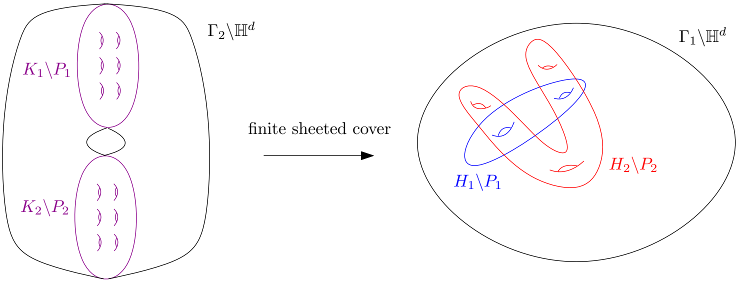
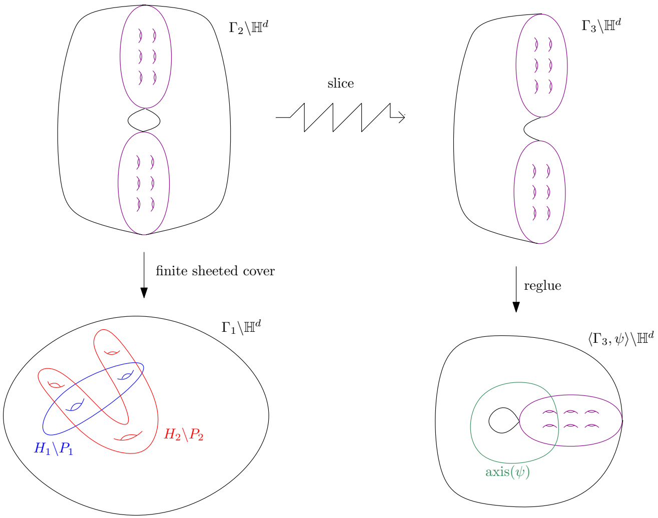


... but since ψ commensurates Γ_1, subgroup separability results by Bergeron-Haglund-Wise (Γ_1 is GFERF) yield a finite index subgroup $\Gamma_2 \leq \Gamma_1$ such that $\mathrm{stab}_{\Gamma_2}(P_1) = K_1$ and $\mathrm{stab}_{\Gamma_2}(P_2) = \psi K_1 \psi^{-1} := K_2$.

Proof of main theorem



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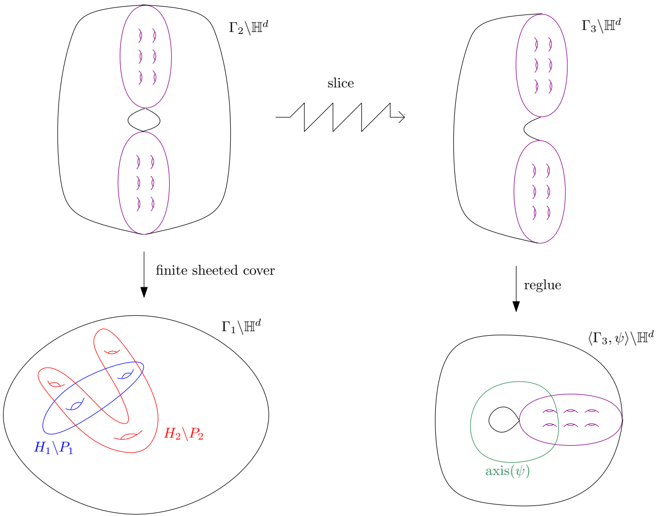

Thus, $\langle \Gamma_3, \psi \rangle$ is a lattice containing ψ. □

Thank you!