ARITHMETIC AND NON-ARITHMETIC LATTICES IN $\mathrm{SO}(d,1)$, GEOMETRICALLY

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0.1. **Hyperboloid models for hyperbolic d-space.** Fix $d \geq 2$ and the quadratic form

$$q_0(x_0, \dots, x_d) = x_0^2 + \cdots + x_{d-1}^2 - x_d^2.$$

The set

$$\mathbb{H}^d := \{x \in \mathbb{R}^{d+1} \mid q_0(x) = -1 \text{ and } x_d > 0\}$$

is the hyperboloid model for d-dimensional hyperbolic space.

The geodesic dimension i geodesic planes in this model are given by non-empty intersections of $\mathbb{H}^d \subseteq \mathbb{R}^{d+1}$ with dimension $i+1$ subspaces of $\mathbb{R}^{d+1}$. For example, if $v \in \mathbb{R}^{d+1} \backslash 0$ and $q_0(v) > 0$, then the subspace $v^\perp \cap \mathbb{H}^d$ is a geodesic hyperplane.

The orientation preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^d)$ can be identified with the group

$$\mathrm{SO}^+(d,1) := \{A \in \mathrm{SL}(d+1,\mathbb{R}) \mid A^t J A = J \text{ and } A(\mathbb{H}^d) = \mathbb{H}^d\}$$

where $J := \mathrm{diag}(1, \dots, 1, -1)$ is the coefficient matrix of q_0.

0.2. **Quadratic forms and other orthogonal groups.** Fix a real quadratic form q of signature $(d,1)$ – that is, there exists $M \in \mathrm{GL}(d+1,\mathbb{R})$ so that $q(Mx) = q_0(x)$ for all $x \in \mathbb{R}^{d+1}$.

Examples: The following are quadratic forms of signature $(d,1)$.

$$q_1(x) = x_0^2 + ... + x_{d-1}^2 - \sqrt{2}x_d^2$$
$$q_2(x) = 3x_0^2 + ... + x_{d-1}^2 - \sqrt{2}x_d^2$$
$$q_3(x) = x_0^2 + ... + x_{d-1}^2 + x_d^2 + \mu x_{d-1}x_d \quad \text{for } \mu > 2 \text{ (exercise)}$$

Let Q be the symmetric coefficient matrix[1] for q. For any subring R of $\mathbb{R}$, the special orthogonal group of q over R is defined by

$$\mathrm{SO}(q,R) := \{A \in \mathrm{SL}(d+1,R) \mid A^t Q A = Q\}.$$

[1]Any quadratic form q can be expressed by a symmetric coefficient matrix $(C + C^t)/2$, where C is any coefficient matrix for q.

and we let $\mathrm{SO}^+(q,R)$ be the index-2 subgroup of $\mathrm{SO}(q,R)$ preserving the two components of $\{x \in \mathbb{R}^{d+1} \mid q(x) = -1\}$. Note that

$$\mathrm{SO}^+(q,R) = M\,\mathrm{SO}^+(q_0,R)M^{-1}.$$

Since $\mathrm{SO}^+(q_0,\mathbb{R}) = \mathrm{SO}^+(d,1)$, we will identify $\mathrm{SO}^+(q,\mathbb{R})$ with $\mathrm{Isom}^+(\mathbb{H}^d)$.

0.3. **Arithmetic lattices of simplest type.** Let k be a totally real number field, and fix q a quadratic form in $d+1$ variables with coefficients in k. Such a pair (k,q) is **admissible** if q has signature $(d,1)$ at the identity embedding of k, and q^σ is positive definite for any other real embedding σ of k.

Examples: $(\mathbb{Q}(\sqrt{2}), q_1)$ and $(\mathbb{Q}(\sqrt{2}), q_2)$ are admissible. $(\mathbb{Q}(\mu), q_3)$ is admissible if $\mu = \lambda + \lambda^{-1}$ for a Salem number[2] λ (exercise).

Non-example: For $q_4(x) = x_0^2 + ... + x_{d-1}^2 - \cos(2\pi/p)x_d^2$ and any prime $p \geq 11$, the pair $(\mathbb{Q}(\cos(2\pi/p)), q_4)$ is not admissible: $k = \cos(2\pi/p)$ is totally real and q_4 has signature $(d,1)$, but if $p \geq 11$ and σ is the Galois embedding of k generated by $\cos(2\pi/p) \mapsto \cos(4\pi/p)$, then q_4^σ also has signature $(d,1)$.

In the theorem below, $\mathcal{O}_k$ is the ring of integers of the number field k, and (q,k) is anisotropic if $q(x) \neq 0$ for all $x \in k^{d+1}$.

Theorem 0.1 (Borel–Harish-Chandra [2])**.** *If (k,q) is admissible and $\Gamma := \mathrm{SO}^+(q,\mathcal{O}_k)$, then $M = \mathbb{H}^d/\Gamma$ has finite volume. Further, if $v \in k^{d+1}$, $q(v) > 0$, and $P_v = v^\perp \cap \mathbb{H}^d$, then $S := P_v/\mathrm{stab}_\Gamma(P_1)$ has finite area. M and S are compact if (q,k) is anisotropic.*

Subgroups $\Gamma_1, \Gamma_2 \leq \mathrm{Isom}^+(\mathbb{H}^d)$ are **commensurable** if $\Gamma_1 \cap \Gamma_2$ has finite index in each Γ_1, Γ_2.

Definition 0.2. A discrete subgroup $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^d)$ (and the quotient $\mathbb{H}^d/\Gamma$) is called **arithmetic of simplest type** if Γ is commensurable to a conjugate of $\mathrm{SO}^+(q,\mathcal{O}_k)$ for some admissible pair (q,k).

Proposition 0.3. *Every hyperbolic d-manifold of simplest type contains infinitely many finite area totally geodesic hypersurfaces.*

This is a corollary of the Borel–Harish-Chandra theorem, since k^{d+1} is dense in $\mathbb{R}^{d+1}$, and this property is commensurability invariant.

0.4. **Non-arithmetic lattices.** An **arithmetic building block** is a component of $M//S$, where M is a hyperbolic d-manifold of simplest type and S is a disjoint union of compact totally geodesic hypersurfaces in M. If a closed

[2]A real algebraic integer is λ is Salem if $\lambda > 1$ and its Galois conjugates consist of $1/\lambda$ and at least one complex value on the unit circle.

hyperbolic manifold is a union of arithmetic building blocks with disjoint interiors, then it is **built from** this collection of building blocks. Arithmetic building blocks are **similar** if they are building blocks for commensurable manifolds; otherwise, the building blocks are *dissimilar*. It follows from work of Gromov–Piatetski-Shapiro [4] that similarity is an equivalence relation.

Theorem 0.4 (Fisher-Lafont-Miller-Stover [3])**.** *If M is a closed hyperbolic d-manifold built from two dissimilar arithmetic building blocks, then the collection of closed totally geodesic hypersurfaces is finite.*

Corollary 0.5. *If M is a closed hyperbolic d-manifold built from two dissimilar building blocks, then M is not arithmetic of simplest type. (In fact, M is also not arithmetic.)*

This corollary follows from the proposition and the FLMS theorem, and was proven by Gromov–Piatetski-Shapiro [4] ealier by simpler methods.

The following is a key lemma for the FLMS theorem.

Lemma 0.6 (Angle rigidity)**.** *For M as in the theorem has arithmetic building blocks N_1, N_2 and T is a closed totally geodesic hypersurface in M, then either T is a component of ∂N_1 or T meets ∂N_1 orthogonally.*

Question: If M is a closed hyperbolic d-manifold built from two *similar* arithmetic building blocks, it is said to be **inbred**. Does the angle rigidity theorem hold for non-arithmetic inbred hyperbolic d-manifolds? (Remark: It is known by Bader-Fisher-Miller-Stover [1] that conclusion of FLMS theorem still holds for non-arithmetic inbred manifolds.)

0.5. **Constructing glue-able dissimilar building blocks.** Idea: The quadratic forms q_1, q_2 above agree on the subspace $x_0 = 0$. The plane

$$P := \mathbb{H}^n \cap \{x_0 = 0\}$$

then projects to isometric orbifolds in $\mathbb{H}^n / \operatorname{SO}^+(q_i, k)$ for $i = 1, 2$ and $k = \mathbb{Q}(\sqrt{2})$. By passing to finite index manifold covers N_1, N_2, we may assume that P projects to closed embedded isometric 2-sided hypersurfaces in N_1, N_2. Cut along these hypersurfaces to construct the building blocks, which are dissimilar.

References

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2. Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535. MR 147566

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4. M. Gromov and I. Piatetski-Shapiro, *Nonarithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 66, 93–103. MR 932135 (89j:22019)